On good EQ-algebras

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Abstract

A special algebra called EQ-algebra has been recently introduced by Vilém Novák. Its original motivation comes from fuzzy type theory, in which the main connective is fuzzy equality. EQ-algebras have three binary operations — meet, multiplication, fuzzy equality – and a unit element. They open the door to an alternative development of fuzzy (many-valued) logic with the basic connective being fuzzy equality instead of implication. This direction is justified by the idea due to G. W. Leibniz that "a fully satisfactory logical calculus must be an equational one".

In this paper, we continue the study of EQ-algebras and their special cases. We introduce and study the prefilters and the filters of separated EQ-algebras. We give great importance to the study of good EQ-algebras. As we shall see in this paper that the "goodness" property (and thus also separateness) is necessary for reasonably behaving algebras. We enrich good EQ-algebras with an unary operation Δ (the so-called Baaz delta) fulfilling some additional assumptions, which is heavily used in fuzzy logic literature. We show that the characterization theorem obtained till now for representable good EQ-algebras hold also for the enriched algebra.

Key words: EQ-algebras, fuzzy equality, fuzzy logic, residuated lattices, BCK-algebras, representable algebras

1. Introduction

The algebraic semantics of fuzzy logic involve various kinds of residuated lattices whose operations generalize the classical boolean truth functions on $\{\mathbf{0}, \mathbf{1}\}$. This means that the truth functions behave classically when restricted to the

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values **0**, **1** (see [14, 21, 22, 23, 41]). A characteristic feature of residuated lattices is that they have two binary operations that can be used as interpretations of conjunction, namely lattice meet and multiplication (a monoidal operation). The latter is then closely tied to residuation (the implication operation) via adjunction. Consequently, fuzzy logics have two conjunctions — meet conjunction and strong conjunction — and implication, which is closely tied to the latter. Recall that when $[0, 1]$ is used as the set of truth values, then the most widely used operations interpreting conjunction are triangular norms (t-norms), which are monotone, commutative, associative binary operations on $[0, 1]$ with neutral element 1 (see [29]). For some applications of fuzzy logic, however, it turns out that one needs more flexibility in the choice of the conjunction: in particular, the commutativity of the strong conjunction may be omitted. The importance of non-commutativity is raised when attempting to model common sense reasoning; for example, a "joyful and clever boy" has not the same meaning as a "clever and joyful boy," since the stress in natural language is generally placed on the first component. Therefore, several papers have appeared on fuzzy logics with non-commutative conjunction (see [1, 11, 16, 24, 25, 26, 32, 33, 34, 35, 31, 50]).

Unlike the above mentioned direction in algebraic semantics, where the essential operations are multiplication and residuation, and the most important operations in the corresponding fuzzy logics are strong conjunction and implication, there is another direction in the development of logic, inspired by G. W. Leibniz's proclamation that "a fully satisfactory logical calculus must be an equational one" (cf. [9]). Therefore, as an alternative to residuated lattices, a special algebra called EQ-algebra has been introduced by V. Novák in [38] and elaborated in [39]. The original motivation was to introduce a special algebra of truth values for fuzzy type theory (FTT) (see, [37]), which generalizes the system of classical type theory (cf. [4]) in which the sole basic connective is equality. Analogously, the basic connective in FTT should be fuzzy equality. Another motivation for EQ-algebras stems from the equational style of proof in logic (cf. [49]).

From the point of view of logic, the main difference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. While in residuated lattices it is obtained from (strong) conjunction, in EQ-algebras it is obtained from equivalence. Consequently, the two kinds of algebras differ in several essential points despite their many similar or identical properties.

EQ-algebra has three binary operations — meet $\wedge$, multiplication $\otimes$, and fuzzy equality $\sim$ – and a unit element, while the *implication* $\rightarrow$ is derived from fuzzy equality $\sim$. Its axioms reflect the "substitution principle" stating that if we replace an object by another one equal to the former then the result is not changed (i.e., in the case of fuzzy equality, it is not "worse" in some reasonable sense). This basic structure in fuzzy logic is *ordering*, represented by $\wedge$-semilattice, with maximal element **1**.

In this paper, we continue the study of EQ-algebras and their special cases, begun in [38, 39] and [13]. As we shall show in this paper, the commutativity axiom of the multiplication originally assumed in [39, Definition 1] is superfluously sever and restrictive, i.e., a weaker requirement put on non-commutative multiplications is sufficient to guarantee all the expected general properties of fuzzy equalities and EQ-algebras. From the point of view of potential applications, it seems very interesting that unlike [24], we can have non-commutativity without necessity to introduce two kinds of implication. Thus, the applications especially in modeling of commonsense reasoning in natural language might be more natural.

In this paper, we give great importance to the study of good EQ-algebras. As we shall see in this paper that the "goodness" property (and thus also separateness) is necessary for reasonably behaving algebras. In particular, we show that $\{\rightarrow, 1\}$-reducts of good EQ-algebras are BCK-algebras. This fact played an important role in our characterization of the representable class of good EQ-algebras (see [13]). We enrich good EQ-algebras with unary operation Δ (the so-called Baaz delta) fulfilling some additional assumptions, which is heavily used in fuzzy logic literature. We show that the characterization theorem obtained till now for representable good EQ-algebras (see [13]) hold also for the enriched algebra.

This paper is organized as follows: In Section 2 we review the basic definitions and properties of EQ-algebras and their special kinds. In Section 3, we continue the study of EQ-algebras and prove several new and important properties of EQ-algebras and their special cases. In Section 4, we introduce and study the prefilters and filters of EQ-algebras. Section 5 is dedicated to the study of good EQ-algebras. In Section 6, we enrich good EQ-algebras with an unary operation Δ. We devote Section 7 to characterize the representable class of the enriched algebras. The results are summarized in Section 8.

2. EQ-algebras: An overview

2.1. Definitions and fundamental properties

Definition 1
An EQ-algebra is an algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

of type $(2, 2, 2, 0)$ *where for all* $a, b, c, d \in E$,

(A1) $\langle E, \wedge, \mathbf{1} \rangle$ is a $\wedge$-semilattice with top element $\mathbf{1}$. We set $a \leq b$ iff $a \wedge b = a$, as usual.

(A2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leq b$.

(A3) $a \sim a = \mathbf{1}$ (reflexivity axiom)

(A4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$ (substitution axiom)

(A5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence axiom)

(A6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity axiom)

(A7) $a \otimes b \leq a \sim b$ (boundedness axiom)

The operation "$\wedge$" is called meet (infimum), "$\otimes$" is called multiplication, and "$\sim$" is called fuzzy equality.

Remark 1
The definition of EQ-algebras in [39, Definition 1] includes another axiom, namely

$$(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c) \tag{1}$$

As we shall see in the next lemma (see Lemma 1 (e)), we do not need this axiom, since it follows from the other axioms. Moreover, Definition 1 differs from the original definition of EQ-algebras (see [39, Definition 1]) in that the multiplication $\otimes$ needs not be commutative. As we shall see in this paper that the commutativity axiom of multiplications is superfluously restrictive, i.e., a weaker requirement put on non-commutative multiplications is sufficient to guarantee all expected general properties of fuzzy equalities and EQ-algebras. Throughout this paper, EQ-algebras with commutative multiplications, i.e. as in [39], will be called commutative EQ-algebras.

Clearly, $\leq$ is the classical partial order. We also set

$$a \to b = (a \wedge b) \sim a, \tag{2}$$

$$\tilde{a} = a \sim \mathbf{1} \tag{3}$$

for $a, b \in E$. The derived operation (2) is called *implication*. Hence, we may rewrite (A6) and (1) as

$$a \to (b \wedge c) \leq a \to b, \tag{4}$$

$$a \to b \leq (a \wedge c) \to b, \tag{5}$$

respectively. If $\mathcal{E}$ also contains a bottom element $\mathbf{0}$, then we may define the unary operation $\neg$ on $\mathcal{E}$ by

$$\neg a = a \sim \mathbf{0}, \qquad a \in E \tag{6}$$

and call $\neg a$ a *negation* of $a \in E$.

The substitution axiom (A4) can be also seen as a special form of extensionality (see, e.g., [23]). Note also that axiom (A6) and (1) , in fact, express the isotonicity of $\to$ w.r.t. the second variable and the antitonicity of $\to$ w.r.t. the first variable.

Note that all of the essential properties of EQ-algebras presented in [39, Section 3]) were proven without using the commutativity of the multiplication $\otimes$ (the only exception is the transitivity of $\sim$ in [39, Theorem 1(b)]), and quite often also without using its associativity. In the following lemma we shall give an alternative proof for the transitivity without using the commutativity axiom of the multiplication.

Lemma 1
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle$ be an EQ-algebra. Then the following properties hold for all a, b, c in E:

(a) $a \sim b = b \sim a$, *(symmetry)*

(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$, *(transitivity)*

(c) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$,

(d) $(a \sim d) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim c)$,

(e) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$.

PROOF: (a) The proof is the same as in [39, Theorem 1 (a)] and uses axioms (A3) and (A4).

(b) By (A4) and item (a) (the symmetry of $\sim$),we have

$$\begin{aligned}(a \sim b) \otimes (b \sim c) = (b \sim a) \otimes (b \sim c) = ((b \wedge \mathbf{1}) \sim a) \otimes (b \sim c)\\ \leq (a \sim (c \wedge \mathbf{1})) = a \sim c.\end{aligned}$$

(c) By (A4), we get
$a \sim d = ((a \wedge b) \sim (a \wedge b)) \otimes (a \sim d) \leq (a \wedge b) \sim (d \wedge b)$.

(d) By properties (b), (c) and the monotonicity of $\otimes$, we get

$$(a \sim d) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim (a \wedge b)) \otimes ((a \wedge b) \sim c) \leq ((d \wedge b) \sim c).$$

(e) By (A3), (A4) and the symmetry of $\sim$, we get
$(a \wedge b) \sim a = ((a \wedge c) \sim (a \wedge c)) \otimes ((a \wedge b) \sim a) \leq (a \wedge b \wedge c) \sim (a \wedge c)$ □

Proving (b), (d) and (e) in the last lemma, we have shown that the definition of EQ-algebras from [39] is practically the same as ours.

Definition 2 ([39])
Let $\mathcal{E}$ be an EQ-algebra. We say that it is

- semiseparated *if for all $a, b \in E$,*

$$a \sim \mathbf{1} = \mathbf{1} \text{ implies } a = \mathbf{1}. \tag{7}$$

- separated *if for all $a, b \in E$,*

$$a \sim b = \mathbf{1} \text{ implies } a = b. \tag{8}$$

- spanned *if it contains a bottom element* $\mathbf{0}$ *and*

$$\tilde{\mathbf{0}} = \mathbf{0} \tag{9}$$

- good *if for all $a \in E$,*

$$a \sim \mathbf{1} = a. \tag{10}$$

- residuated *if for all* $a, b, c \in E$,

$$(a \otimes b) \wedge c = a \otimes b \quad \text{iff} \quad a \wedge ((b \wedge c) \sim b) = a. \tag{11}$$

- involutive *(IEQ-algebra) if for all* $a \in E$,

$$\neg\neg a = a. \tag{12}$$

- lattice-ordered EQ-algebra *if it has a lattice reduct*[1]*;*

- *and* lattice EQ-algebra *(an* ℓEQ-algebra*) if it is a lattice-ordered EQ-algebra in which the following substitution axiom holds for all* $a, b, c, d \in E$*:*

$$((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c). \tag{13}$$

Note that an EQ-algebra can be lattice-ordered but not necessarily an ℓEQ-algebra. Obviously, each separated EQ-algebra is semiseparated. If the EQ-algebra is good then it is spanned, but note vice-versa. Clearly, (11) can be written classically as $a \otimes b \leq c$ iff $a \leq b \to c$.

We list below some properties of EQ-algebras from [39] that will be used in the paper:

Lemma 2
The following properties hold in all EQ-algebras, for all $a, b, c \in E$*:*

(a) $a \otimes b \leq a \wedge b \leq a,\ b$ and $b \otimes a \leq a \wedge b \leq a,\ b$,

(b) $a = b$ *implies* $a \sim b = \mathbf{1}$,

(c) $(a \to b) \otimes (b \to c) \leq a \to c$, *(transitivity of implication)*

(d) $b \leq \tilde{b} \leq a \to b$,

(e) $a \sim b \leq a \to b$ *and* $a \to a = \mathbf{1}$, *(i.e.,* $\to$ *is reflexive)*

(f) $(a \to b) \otimes (b \to a) \leq a \sim b \leq (a \to b) \wedge (b \to a)$*. If* $\mathcal{E}$ *is linearly ordered then* $\leq$ *can be replaced by equality.*

(g) Let $a \leq b$*. Then*

$$a \to b = \mathbf{1}, \quad a \sim b = b \to a, \quad \tilde{a} \leq \tilde{b}, \quad c \to a \leq c \to b \text{ and } b \to c \leq a \to c.$$

[1]Given an algebra $\langle E, F \rangle$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $\langle E, F' \rangle$ is called the F'-reduct of $\langle E, F \rangle$. The subalgebras of $\langle E, F' \rangle$ are then called F'-subreducts of $\langle E, F \rangle$.

By Lemma 2 (c), (e) and (f), the implication $\to$ is a fuzzy ordering w.r.t. the fuzzy equality $\sim$ (this notion was studied extensively by Bodenhofer [7]). As mentioned in [39], we can regard an EQ-algebra as a set endowed with a classical partial order $\leq$ (and the corresponding classical equality $=$), and a top element $\mathbf{1}$ and the fuzzy equality $\sim$ together with a fuzzy ordering $\to$.

The following lemma characterizes a compatibility of $\sim$ with the ordering $\leq$, namely that if the "distance" between elements increases (in the sense of $\leq$), then the degree of their equality decreases.

Lemma 3 ([39])
If $a \leq b \leq c$, then $c \sim a \leq c \sim b$ as well as $a \sim c \leq a \sim b$.

Lemma 4 ([39])
If $a \to b = \mathbf{1}$, then $a \leq b$ or $a \sim b = \mathbf{1}$ or $a \| b$ (i.e., a, b are incomparable).

According to this simple lemma, it may happen that $a \to b = \mathbf{1}$, $a \sim b < \mathbf{1}$ and $a \| b$. Another consequence is that we can have comparable elements a, b such that $a > b$, $a \sim b = \mathbf{1}$ and $a \to b = \mathbf{1}$. Such an ordered couple $\langle a, b \rangle$ will be called *pathological*. An EQ-algebra that does not contain pathological couples is called *regular*.

Lemma 5 ([39])
(a) In every good EQ-algebra, the following inequality holds for all $a, b \in E$:

$$a \leq (a \sim b) \sim b.$$

(b) A good EQ-algebra is separated, and thus does not contain pathological couples, i.e., it is regular.

(c) Each residuated EQ-algebra is good (and thus separated).

An EQ-algebra $\mathcal{E}$ is *complete* if it is a complete $\wedge$-semilattice. Obviously, a complete EQ-algebra is a complete lattice (see, e.g., [6]) but not necessarily an ℓEQ-algebra. Every finite EQ-algebra is lattice-ordered.

2.2. Examples of EQ-algebras

In this section, we give a few examples of EQ-algebras.

Example 1
Consider $E = \{\mathbf{0}, a, b, c, \mathbf{1}\}$ to be a five-element chain. The following multiplication and the fuzzy equality define a linearly ordered EQ-algebra that is not residuated:

$\otimes$	$\mathbf{0}$	a	b	c	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a
b	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	b	b
c	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	c	c
$\mathbf{1}$	$\mathbf{0}$	a	b	c	$\mathbf{1}$

$\sim$	$\mathbf{0}$	a	b	c	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	a	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	a	$\mathbf{1}$	a	a	a
b	$\mathbf{0}$	a	$\mathbf{1}$	b	b
c	$\mathbf{0}$	a	b	$\mathbf{1}$	c
$\mathbf{1}$	$\mathbf{0}$	a	b	c	$\mathbf{1}$

Note that the multiplication $\otimes$ is non-commutative, but is associative. This algebra is good.

For more examples of non-trivial finite EQ-algebras, including linearly ordered ones, see [39].

Let $\Lambda = (E, \wedge, \vee, \otimes, \Rightarrow, \mathbf{1})$ be a residuated lattice (the definition and several useful properties of residuated lattices can be found in [18]). The following two kinds of biresiduation operations can be introduced:

$$a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a) \tag{14}$$

$$a \overset{\circ}{\Leftrightarrow} b = (a \Rightarrow b) \otimes (b \Rightarrow a). \tag{15}$$

Both operations are natural interpretations of equivalence, since they are reflexive, symmetric, and transitive in the following sense:

$$(a \boxdot b) \otimes (b \boxdot c) \leq a \boxdot c$$

for all $a, b, c \in E$, where $\boxdot \in \{\Leftrightarrow, \overset{\circ}{\Leftrightarrow}\}$. Note that the biresiduation operation (14) has been used in the development of IMTL-FTT (see [37]).

Example 2
Let $\Lambda = \langle E, \wedge, \vee, \otimes, \Rightarrow, \mathbf{1}\rangle$ be a residuated lattice.

(i) The algebra $\mathcal{E}_\Lambda = \langle E, \wedge, \otimes, \Leftrightarrow, \mathbf{1}\rangle$ is a residuated EQ-algebra. If $\mathcal{E}$ is linearly ordered, then $\mathcal{E}'_\Lambda = \langle E, \wedge, \otimes, \overset{\circ}{\Leftrightarrow}, \mathbf{1}\rangle$ is also a residuated EQ-algebra (since both $\Leftrightarrow$ and $\overset{\circ}{\Leftrightarrow}$ coincide; cf. [39]).

(ii) Let $\odot \leq \otimes$ be an isotone monoidal operation on E. Then both $\mathcal{E}^{\#} = \langle E, \wedge, \odot, \Leftrightarrow, \mathbf{1}\rangle$ as well as $\mathcal{E}^{\#\#} = \langle E, \wedge, \bar{\odot}, \Leftrightarrow, \mathbf{1}\rangle$ are a good (and hence separated) EQ-algebras in which $\odot$ and $\bar{\odot}$ are associative, but not necessarily commutative, where $\bar{\odot}$ is the reverse of $\odot$ (defined by $a \bar{\odot} b = b \odot a$). If $\odot < \otimes$, then $\mathcal{E}^{\#}$ is not residuated, because we can have $a \odot b \leq c < a \otimes b$.

(iii) Let (m, n) be a pair of weak negations on E (i.e., they are order-reversing and satisfy, for all a, the relations $a \leq m(n(a))$, $a \leq n(m(a))$ and $n(\mathbf{1}) = m(\mathbf{1}) = \mathbf{0}$ [19]). Define $\odot$ on E by

$$a \odot b = \begin{cases} \mathbf{0} & a \leq m(b) \\ a \otimes b & \textit{otherwise}, \end{cases} = \begin{cases} \mathbf{0} & b \leq n(a) \\ a \otimes b & \textit{otherwise}. \end{cases}$$

It is easy to see that $\odot \leq \otimes$. Also, $\odot$ need not be commutative nor associative. Moreover, if the pair of weak negations (m, n) is compatible with $\otimes$ (see [19]), then $\odot$ is associative, and hence $\mathcal{E}' = \langle E, \wedge, \odot, \Leftrightarrow, \mathbf{1}\rangle$ is an (noncommutative) EQ-algebra. Finally, if $n = m$, then $\odot$ must be commutative.

Note that the EQ-algebras in Example 2 (ii)–(iii) are non-residuated (even if the multiplications are commutative). The following example is an instance of Example 2.

Example 3
Let $E = [0,1]$ and define the product $\otimes$ and residuum $\Rightarrow$ on E as follows:

$$a \otimes b = \begin{cases} \mathbf{0}, & a+b \le \frac{1}{2} \\ \min\{a,b\}, & a+b > \frac{1}{2}, \end{cases} \qquad a \Rightarrow c \quad = \begin{cases} 1, & a \le c \\ \max\{\frac{1}{2}-a, c\}, & a > c. \end{cases}$$

Then $\Lambda = \langle E, \wedge, \vee, \otimes, \Rightarrow, 0, 1\rangle$ is a residuated lattice.

Define $\odot$ and $\bar{\odot}$ on $[0,1]$ by (see [32]):

$$a \odot b = \begin{cases} \mathbf{0}, & 2a+b \le 1 \\ \min\{a,b\}, & 2a+b > 1, \end{cases} \qquad a \bar{\odot} b = \begin{cases} \mathbf{0}, & a+2b \le 1 \\ \min\{a,b\}, & a+2b > 1. \end{cases}$$

Both $\odot$ and $\bar{\odot}$ are isotone monoidal operations on $[0,1]$, but they are not commutative. It can be verified directly that $\odot$, $\bar{\odot} \le \otimes$, and

$$a \Leftrightarrow c = a \overset{\circ}{\Leftrightarrow} c = \begin{cases} 1, & a = c \\ \max(\frac{1}{2}-a, c), & a > c \\ \max(\frac{1}{2}-c, a), & a < c. \end{cases}$$

Hence, $\mathcal{E} = \langle E, \wedge, \otimes, \Leftrightarrow, 1\rangle$ is a residuated EQ-algebra, while both $\mathcal{E}' = \langle E, \wedge, \odot, \Leftrightarrow, 1\rangle$ and $\mathcal{E}'' = \langle E, \wedge, \bar{\odot}, \Leftrightarrow, 1\rangle$ are noncomutative EQ-algebras that are not residuated but are good.

Let $* : [0,1]^2 \to [0,1]$ be the nilpotent minimum (see Fodor [17]) given by

$$a * b = \begin{cases} \mathbf{0}, & a+b \le 1 \\ \min\{a,b\}, & a+b > 1. \end{cases}$$

One can see that $* \le \otimes$, so $\mathcal{E}^{\#} = \langle E, \wedge, *, \Leftrightarrow, 1\rangle$ is a good (and hence separated) commutative EQ-algebra.

2.3. Prelinear good EQ-algebras

Definition 3 ([13])
An EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is said to be prelinear if for all $a, b \in E$, 1 is the unique upper bound in E of the set $\{(a \to b), (b \to a)\}$.

Note that the prelinearity does not necessitate the presence of a join operator in E^2. However, every prelinear and good EQ-algebra is a lattice-ordered whereby the join operation is definable in terms of the meet $\wedge$ and the implication $\to$ operations as the following theorem asserts:

[2] This approach is well known in literature, see e.g., Abdel-Hamid and Morsi [2] where the authors established a representation theorem of prelinear residuated algebras, in which the lattice structure is not assummed.

Theorem 1 ([13])
Every prelinear and good EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is a prelinear and good ℓEQ-algebra, whereby the join operation is given by

$$a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a), \qquad a, b \in E \tag{16}$$

We know that the underlying poset E, of an EQ-algebra $\mathcal{E}$ need not be a join-semilattice. Nevertheless, given $a, b \in E$, we shall write $a \vee b = 1$ meaning that the supremum of $\{a, b\}$ in E, exists and is equal to 1.

Proposition 1 ([13])
The following are equivalent in each good EQ-algebra $\mathcal{E}$, for all $a, b, c, d \in E$:

(i) $\mathcal{E}$ is prelinear and satisfies the quasi-identity

$$a \vee b = 1 \textit{ implies } a \vee (d \to (d \otimes (c \to (b \otimes c)))) = 1 \tag{17}$$

(ii) $\mathcal{E}$ satisfies the identity

$$(a \to b) \vee (d \to (d \otimes (c \to ((b \to a) \otimes c)))) = 1 \tag{18}$$

(iii) $\mathcal{E}$ satisfies

$$(a \to b) \to u \leq [(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u] \to u \tag{19}$$

(iv) $\mathcal{E}$ satisfies

$$(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u \leq ((a \to b) \to u) \to u \tag{20}$$

Proposition 2 ([13])
The following properties are equivalent in each good and commutative EQ-algebra $\mathcal{E}$, for all $a, b, c, u \in E$:

(i) $\mathcal{E}$ is prelinear and satisfies the quasi-identity (17).

(ii) $\mathcal{E}$ is prelinear and satisfies the quasi-identity

$$a \vee b = 1 \textit{ implies } a \vee (c \to (b \otimes c)) = 1 \tag{21}$$

(iii) $\mathcal{E}$ satisfies the identity

$$(a \to b) \vee (c \to ((b \to a) \otimes c)) = 1 \tag{22}$$

(iv) $\mathcal{E}$ satisfies

$$(c \to ((b \to a) \otimes c)) \to u \leq ((a \to b) \to u) \to u \tag{23}$$

Prelinearity alone does not characterize representable good (commutative) EQ-algebras (see [13, Example 4]). El-Zekey [13] has proved that representable good EQ-algebras can be characterized by (20). Moreover, By Proposition 2, if the multiplication $\otimes$ is commutative then the inequality (20) is equivalent to (23). Consequently, the representable good and commutative EQ-algebras can be characterized by (23).

3. New properties of EQ-algebras

We have the following new properties of EQ-algebras.

Theorem 2
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$ be an EQ-algebra. Define the reverse $\bar{\otimes}$ of $\otimes$ by $a \bar{\otimes} b = b \otimes a$. Then $\bar{\mathcal{E}} = \langle E, \wedge, \bar{\otimes}, \sim, \mathbf{1} \rangle$ is an EQ-algebra.

PROOF: It is sufficient to note that, by Lemma 1 (d) and by axioms (A5) and (A7) with the symmetry of $\sim$, $\bar{\mathcal{E}} = \langle E, \wedge, \bar{\otimes}, \sim, \mathbf{1} \rangle$ satisfies axioms ((A1)–(A7)). $\square$

Theorem 3
Let Ψ be an inequality in the language of EQ-algebras. Let $\bar{\Psi}$ be an inequality obtained from Ψ by exchanging the multiplication $\otimes$ with its reverse $\bar{\otimes}$ in all of its occurrences in Ψ. Then Ψ is universally valid if, and only if, $\bar{\Psi}$ is.

PROOF: The result follows from Theorem 2 noting that $\bar{\bar{\otimes}} = \otimes$. $\square$

Theorem 4
The class of EQ-algebras is a variety.

PROOF: Just note that the isotonicity of $\otimes$ in the first argument (i.e., $a \leq b$ implies $a \otimes c \leq b \otimes c$ for all $c \in E$) is equivalent to the identity $((a \wedge b) \otimes c) \wedge (b \otimes c) = (a \wedge b) \otimes c$, and likewise for the second argument. All the other properties stated in Definition 1 can be expressed using equations. $\square$

Lemma 6
The following properties hold in all EQ-algebras:

(a) $(a \sim b) \otimes (c \sim d) \leq (a \wedge c) \sim (b \wedge d)$,

(b) $a \sim d \leq ((a \wedge b) \sim c) \sim ((d \wedge b) \sim c)$,

(c) $a \sim d \leq (a \sim c) \sim (d \sim c)$,

(d) $a \sim d \leq (b \to a) \sim (b \to d)$,

(e) $a \to d \leq (b \to a) \to (b \to d)$,

(f) $b \to a \leq (a \to d) \to (b \to d)$,

(g) $a \otimes (a \sim b) \leq \tilde{b}$ and $(a \sim b) \otimes a \leq \tilde{b}$,

(h) $a \otimes (a \to b) \leq \tilde{b}$ and $(a \to b) \otimes a \leq \tilde{b}$, *(weak modus ponens)*

(i) $a \leq b \sim c$ *implies* $a \otimes b \leq \tilde{c}$ and $b \otimes a \leq \tilde{c}$,

(j) $a \leq b \to c$ *implies* $a \otimes b \leq \tilde{c}$ and $b \otimes a \leq \tilde{c}$,

(k) $(b \to c) \otimes (a \to b) \le a \to c$.

PROOF: (a) By Lemma 1 (c), we have $a \sim b \le (a \wedge c) \sim (b \wedge c)$ and $c \sim d \le (b \wedge c) \sim (b \wedge d)$. Hence, by the order properties of $\otimes$ and transitivity of $\sim$ (Lemma 1 (b)), we get

$$(a \sim b) \otimes (c \sim d) \le ((a \wedge c) \sim (b \wedge c)) \otimes ((b \wedge c) \sim (b \wedge d)) \le (a \wedge c) \sim (b \wedge d).$$

(b) By Lemma 1 (c), we have

$$a \sim d \le (a \wedge b) \sim (d \wedge b) = ((a \wedge b) \sim (d \wedge b)) \otimes (c \sim c) \le \\ ((a \wedge b) \sim c) \sim ((d \wedge b) \sim c)$$

(by E5).

(c) is obtained from (b) by setting $b = \mathbf{1}$.

(d) is obtained from (b) by setting $c = b$.

(e) From (d), we get

$$a \to d = a \sim (a \wedge d) \le (b \to a) \sim (b \to a \wedge d) \le \\ (b \to a) \to (b \to a \wedge d) \le (b \to a) \to (b \to d)$$

(by monotonicity of $\to$).

(f) From (c), (A6), (1) and the hybrid monotonicity properties of $\to$, we get the following chain of inequalities:

$$b \to a = (a \wedge b) \sim b \le ((a \wedge b) \sim ((a \wedge b) \wedge d)) \sim (b \sim ((a \wedge d) \wedge b)) \le \\ (a \sim (a \wedge d)) \to (b \sim (b \wedge d)) = (a \to d) \to (b \to d).$$

(g) and (h): the first inequality was proven in [39]; the second inequality follows from the first one by Theorem 3.

(i) and (j) are obtained from (g) and (h), respectively, using the monotonicity of $\otimes$.

(k) is obtained from Lemma 2 (c) using Theorem 3. □

The following lemma is a direct consequence of the results of Lemma 6.

Lemma 7

Let $\mathcal{E}$ be an EQ-algebra with bottom element $\mathbf{0}$. The following hold for all $a, b, c \in E$:

(a) $a \to b \le \neg b \to \neg a$. *Moreover, if $\mathcal{E}$ is involutive, then* $a \to b = \neg b \to \neg a$.

(b) $a \sim b \le \neg a \sim \neg b$. *Moreover, if $\mathcal{E}$ is involutive, then* $a \sim b = \neg a \sim \neg b$.

(c) $(a \sim b) \otimes \neg b \le \neg a$ *and* $\neg b \otimes (a \sim b) \le \neg a$.

(d) $(a \to b) \otimes \neg b \le \neg a$ *and* $\neg b \otimes (a \to b) \le \neg a$.

(e) $\neg b \le (b \to c)$.

(f) $a \sim d \leq \neg(a \wedge b) \sim \neg(d \wedge b)$.

We say that the *multiplication* $\otimes$ *is* $\rightarrow$*-isotone* if

$$a \rightarrow b = \mathbf{1} \text{ implies } (a \otimes c) \rightarrow (b \otimes c) = \mathbf{1}$$

for all $a, b, c \in E$.

Lemma 8
The following properties hold in EQ-algebras:

(a) $a \rightarrow (b \rightarrow a) = \mathbf{1}$.

(b) $(c \rightarrow a) \otimes (c \rightarrow b) \leq c \rightarrow (a \wedge b)$.

(c) Let $a \rightarrow b = \mathbf{1}$ *and* $c \rightarrow d = \mathbf{1}$. *Then* $(a \wedge c) \rightarrow (b \wedge d) = \mathbf{1}$.

(d) If $a \sim b = \mathbf{1}$ *and* $c \sim d = \mathbf{1}$, *then* $(a \wedge c) \sim (b \wedge d) = \mathbf{1}$.

(e) If $a \sim b = \mathbf{1}$, *then* $(a \wedge c) \sim (b \wedge c) = \mathbf{1}$ *and* $(a \sim c) \sim (b \sim c) = \mathbf{1}$.

(f) If $\otimes$ *is* $\rightarrow$*-isotone and* $a \sim b = \mathbf{1}$, *then* $(a \otimes c) \sim (b \otimes c) = \mathbf{1}$.

PROOF: (a) Since $a \leq b \rightarrow a$, we have $a \wedge (b \rightarrow a) \sim a = a \sim a = \mathbf{1}$.
(b) Since $c \rightarrow a = (c \wedge a) \sim c$, using Axiom (A4) we have

$$((c \wedge a) \sim c) \otimes ((c \wedge b) \sim c) \leq (c \wedge (a \wedge b) \sim c) = c \rightarrow (a \wedge b).$$

(c) From $(a \wedge c) \rightarrow a = \mathbf{1}$ and the assumption, we obtain $(a \wedge c) \rightarrow b = \mathbf{1}$ as well as $(a \wedge c) \rightarrow d = \mathbf{1}$. The required inequality then follows from (b).
(d) This follows directly from Lemma 6 (a) and the assumption.
(e) This follows directly from Lemma 1 (c), Lemma 6 (c) and the assumption.
(f) This is a consequence of Lemma 2 (f) and the assumption. □

Note that, in general, the $\rightarrow$-isotonicity of the multiplication $\otimes$ is not provable.

Lemma 9
Let $\mathcal{E}$ *be an EQ-algebra.*

(a) Let $a \leq b \leq c$. *Then* $a \sim b = \mathbf{1}$ *implies that* $a \sim c = b \sim c$ *and* $b \sim c = \mathbf{1}$ *implies* $a \sim c = a \sim b$.

(b) Let $a \sim b = \mathbf{1}$. *Then* $\tilde{a} \sim \tilde{b} = \mathbf{1}$. *If* $a \leq b$, *then* $\tilde{a} = \tilde{b}$.

(c) If $\tilde{a} = \tilde{b} = \mathbf{1}$, *then* $a \sim b = \mathbf{1}$. *If* $a \sim b = \mathbf{1}$ *and* $\tilde{a} = \mathbf{1}$, *then* $\tilde{b} = \mathbf{1}$.

PROOF: (a) $a \sim c \leq b \sim c$ follows from Lemma 3. Furthermore, by the transitivity of $\sim$ (Lemma 1 (b)) and the assumption, we have $(a \sim b) \otimes (b \sim c) = (b \sim c) \leq a \sim c$. The proof of the second part is the same.
(b) The first statement follows from $(a \sim b) \otimes (\mathbf{1} \sim \mathbf{1}) = \mathbf{1} \leq (a \sim \mathbf{1}) \sim (b \sim \mathbf{1})$. The second property follows from Lemma 3 and (a).
(c) This property follows from the previous properties. □

Lemma 10
The following properties hold for all $a, b, c, d \in E$ in every EQ-algebra $\mathcal{E}$:

(a) $((a \sim b) \sim c) \otimes (a \sim d) \leq (d \sim b) \sim c$.

(b) $(a \sim d) \otimes ((a \sim b) \sim c) \leq (d \sim b) \sim c$.

(c) $(a \to b) \otimes (a \to c) \leq a \to b \wedge c$.

(d) $a \to b \leq (a \wedge c) \to (b \wedge c)$.

(e) $a \to b = a \to (a \wedge b)$.

PROOF: (a) From Lemma 6 (c) and the monotonicity of $\otimes$, we get
$((a \sim b) \sim c) \otimes (a \sim d) \leq ((a \sim b) \sim c) \otimes ((a \sim b) \sim (d \sim b)) \leq (d \sim b) \sim c$
(by the transitivity of $\sim$).

(b) This follows from (a) using Theorem 3.

(c) By (2) and Axiom (A4), we have

$$(a \to b) \otimes (a \to c) = ((a \wedge b) \sim a) \otimes ((a \wedge c) \sim a) \leq$$
$$(a \wedge b \wedge c) \sim a = a \to (b \wedge c).$$

(d) From Lemma 6 (a), we get

$$a \to b = (a \sim a \wedge b) \otimes (c \sim c) \leq (a \wedge c) \sim (a \wedge b \wedge c) \leq (a \wedge c) \to (b \wedge c).$$

(e) This property follows from (2) and Lemma 2(g). □

Proposition 3
The following statements are equivalent

(a) A EQ-algebra $\mathcal{E}$ is separated;

(b) $a \leq b$ iff $a \to b = \mathbf{1}$ for all $a, b \in E$.

PROOF: (a) implies (b) is proven in [39]. Conversely, let (b) hold and assume that $a \sim b = \mathbf{1}$. By the properties of $\to$ we have $a \sim b \leq a \to b$ and $a \sim b \leq b \to a$. Thus, $a \to b = \mathbf{1}$ and $b \to a = \mathbf{1}$, that is, $a \leq b$ and $b \leq a$, which gives $a = b$. □

This means that the implication operation $\to$ in a separated EQ-algebra precisely reflects the ordering $\leq$, so the multiplication $\otimes$ is $\to$-isotone in it.

Proposition 4
The following properties are equivalent in every lattice-ordered EQ-algebra $\mathcal{E}$:

(a) $\mathcal{E}$ is ℓEQ-algebra;

(b) $\mathcal{E}$ satisfies, for all a, b, c in E

$$a \sim b \leq (a \vee c) \sim (b \vee c).$$

PROOF: (a) implies (b): From (13), we get

$$a \sim d = ((a \vee b) \sim (a \vee b)) \otimes (a \sim d) \leq (a \vee b) \sim (d \vee b).$$

(b) implies (a): Assume (b). We need only to show that $\mathcal{E}$ satisfies (13). By the assumption and the monotonicity of $\otimes$, we have

$$((a \vee b) \sim c) \otimes (d \sim a) \leq ((a \vee b) \sim c) \otimes ((a \vee b) \sim (b \vee d)) \leq ((d \vee b) \sim c)$$

by the symmetry and transitivity of $\sim$. □

Lemma 11
The following properties hold in every ℓEQ-algebra:

(a) $a \to b = (a \vee b) \to b = (a \vee b) \sim b$;

(b) $(a \sim d) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim c)$;

(c) $(a \sim b) \otimes (c \sim d) \leq (a \vee c) \sim (b \vee d)$;

(d) $a \to b \leq (a \vee c) \to (b \vee c)$.

PROOF: (a) It suffices to note that, by Lemma 2 (g), $(a \vee b) \sim b = (a \vee b) \to b$. The rest is the same as in [39, Proposition 6].

(b) From Proposition 4 (b), the transitivity of $\sim$ and the monotonicity of $\otimes$, we get

$$(a \sim d) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim (a \vee b)) \otimes ((a \vee b) \sim c) \leq ((d \vee b) \sim c).$$

(c) By (a), we have $a \sim b \leq (a \vee c) \sim (b \vee c)$ and $c \sim d \leq (b \vee c) \sim (b \vee d)$. Hence, by the monotonicity of $\otimes$ and transitivity of $\sim$ (Lemma 1(b)), we get

$$(a \sim b) \otimes (c \sim d) \leq ((a \vee c) \sim (b \vee c)) \otimes ((b \vee c) \sim (b \vee d)) \leq (a \vee c) \sim (b \vee d).$$

(d) By item (a), Proposition 4 (b) and the order properties of $\to$, we have

$$a \to b = (a \vee b) \sim b \leq ((a \vee b) \vee c) \sim (b \vee c) \leq (b \vee (a \vee c)) \to (b \vee c) \leq (a \vee c) \to (b \vee c).$$

□

Lemma 12
Let $\mathcal{E}$ be a semiseparated EQ-algebra.

(a) If $a = \mathbf{1}$ and $a \to b = \mathbf{1}$, then $b = \mathbf{1}$.

(b) $\mathcal{E}$ contains no pathological couple $\langle \mathbf{1}, a \rangle$. On the other hand, each regular EQ-algebra is semiseparated.

PROOF: (a) This follows from Lemma 6 (h) and semi-separateness.

(b) The statement is obvious. □

Note that the property (a) is an algebraic counterpart of the classical rule of modus ponens.

4. Filters in EQ-algebras

In this section, we will introduce the concepts of a prefilter and a filter in EQ-algebras and prove some basic facts. The filter and congruence theory of EQ-algebras is quite subtle and still requires more investigation. Various other results can be found in [13]. As emphasized before, EQ-algebras behave differently than residuated lattices. One of the reasons for this different behavior is that $a \to b = \mathbf{1}$ does not imply that $a \leq b$. There are even EQ-algebras that have no proper filter. Therefore, our study of filters in this paper will be *restricted to separated EQ-algebras* only. Some of the results extend to all EQ-algebras provided that the filters in concern exist.

Let us recall some general terminology. An n-ary operation f is *compatible* with a binary relation θ if for all $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ we have

$$f(a_1, \dots, a_n)\, \theta\, f(b_1, \dots, b_n).$$

An equivalence relation θ on (the underlying set of) an algebra $\mathcal{E}$ is a *congruence* if each fundamental operation (and hence each term-definable operation) of $\mathcal{E}$ is compatible with θ. The equivalence class of a is denoted by $[a]_\theta$, and the quotient algebra of $\mathcal{E}$ with respect to θ is denoted by $\mathcal{E}|\theta$. The set of all congruences of $\mathcal{E}$ is denoted by $\mathbf{Con}(\mathcal{E})$.

Let $\mathcal{E}$, $\mathcal{E}'$ be EQ-algebras. A function $q : E \longrightarrow E'$ is a *homomorphism* if

$$q(a \boxdot b) = q(a) \boxdot' q(b),$$

where $\boxdot \in \{\wedge, \otimes, \sim\}$ in $\mathcal{E}$ and $\boxdot' \in \{\wedge', \otimes', \sim'\}$ in $\mathcal{E}'$. The top element and the order are clearly stable under homomorphism, because they are defined using meet.

The following theorem can be proved in a standard way.

Theorem 5
Let θ be a congruence on a (good) EQ-algebra $\mathcal{E}$. Then the factor algebra $\mathcal{E}|\theta$ is a (good, and hence separated) EQ-algebra, and the mapping $q : E \longrightarrow E|\theta$ defined by $q(a) = [a]_\theta$ is a homomorphism.

Note that if $\mathcal{E}$ is a separated EQ-algebra, then the algebra $\mathcal{E}|\theta$ is not, in general, separated. Given a separated EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \rangle$, we shall say that $\theta \in \mathbf{Con}(\mathcal{E})$ is a *relative congruence* of $\mathcal{E}$ if the quotient algebra $\mathcal{E}|\theta$ is a separated EQ-algebra. Note that the trivial congruence is a relative congruence. Also note that in a good EQ-algebra (since good EQ-algebra is a variety), any congruence is a relative congruence.

Definition 4
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle$ be a separated EQ-algebra. A subset $F \subseteq E$ is called a prefilter *of $\mathcal{E}$ if for all $a, b, c \in E$,*

(i) $\mathbf{1} \in F$;

(ii) *If $a, a \to b \in F$ then $b \in F$.*

A prefilter F is said to be a filter *if for all $a, b, c \in E$ we find that $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$. A prefilter F is said to be a* prime prefilter *(or simply* prime*) if for all $a, b \in E$ we find that $a \to b \in F$ or $b \to a \in F$.*

As usual, a prefilter F is called *proper* if $F \neq E$. If $\mathbf{0} \in E$ then a prefilter $F \subset E$ is proper iff $\mathbf{0} \notin F$. A *minimal prime prefilter* is a prime prefilter that does not properly contain any other prime prefilter.

It is easy to see that the singleton $\{\mathbf{1}\}$ is a filter in any separated EQ-algebra, and it is contained in any other filter.

Lemma 13
Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$. The following holds for all $a, b \in E$:

(a) If $a \in F$ and $a \leq b$ then $b \in F$;

(b) If $a, a \sim b \in F$ then $b \in F$;

(c) If $a, b \in F$ then $a \wedge b \in F$;

(d) If $a \sim b \in F$ and $b \sim c \in F$ then $a \sim c \in F$;

(e) $\mathbf{1} \sim b \in F$ iff $b \in F$;

(f) $F = \{b \in E \mid b \sim \mathbf{1} \in F\}$.

PROOF: (a) From Lemma 2 (g) it follows that $a \to b = \mathbf{1}$. The properties (i) and (ii) of a prefilter then imply that $b \in F$.

(b) Due to Lemma 2 (f), it holds that $a \sim b \leq a \to b$. From item (a) it then follows that $a \to b \in F$, so the property (ii) of a prefilter implies that $b \in F$.

(c) From Lemma 2 (a) and Lemma 10 (e), it follows that $b \leq a \to b = a \to a \wedge b$. From item (a), it then follows that $a \to a \wedge b$ and hence, by the property (ii) of a prefilter, $a \wedge b \in F$.

(d) From Lemma 6 (c) and item (a), it follows that $(b \sim c) \sim (a \sim c) \in F$. From item (b), it then follows that $a \sim c \in F$.

(e) Assume $b \in F$. Hence, by item (a) and Lemma 2 (a), $b \leq b \sim \mathbf{1} \in F$. Conversely, assume that $b \sim \mathbf{1} \in F$. From item (b) and the property (i) of a prefilter, it then follows that $b \in F$.

(f) From item (e), it follows that $a \in F$ iff $a \in \{b \in E \mid b \sim \mathbf{1} \in F\}$, that is, that $F = \{b \in E \mid b \sim \mathbf{1} \in F\}$. □

Lemma 14
Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$, $a \sim b \in F$ and $a' \sim b' \in F$. Then the following hold:

(a) $(a \wedge a') \sim (b \wedge b') \in F$;

(b) $(a \sim a') \sim (b \sim b') \in F$;

(c) $(a \to a') \sim (b \to b') \in F$;

(d) *If* $\mathcal{E}$ *is a* ℓ*EQ-algebra, then* $(a \vee a') \sim (b \vee b') \in F$.

PROOF: (a) By Lemma 1 (c) and Lemma 13 (a), it follows that $(a \wedge a') \sim (b \wedge a') \in F$ and $(b \wedge a') \sim (b \wedge b') \in F$. From Lemma 13 (d), it then follows that $(a \wedge a') \sim (b \wedge b') \in F$.

(b) and (d): These two properties have similar proofs, using Lemma 6 (c) and Proposition 4(b), respectively.

(c) This follows immediately using (2) and items (a) and (b). □

Lemma 15
Let F *be a filter of a separated EQ-algebra* $\mathcal{E}$. *For all* $a, b \in E$,

(a) *If* $a, b \in F$ *then* $a \otimes b \in F$;

(b) $a \sim b \in F$ *iff* $a \leftrightarrow b \in F$ *iff "*$a \to b \in F$ *and* $b \to a \in F$*" iff* $a \overset{\circ}{\leftrightarrow} b \in F$;

(c) *If* $a \sim b \in F$, *then* $(a \otimes c) \sim (b \otimes c) \in F$ *and* $(c \otimes a) \sim (c \otimes b) \in F$ *for all* $c \in E$.

PROOF: (a) From Lemma 2 (a) and Lemma 13 (a), it follows that $b \leq \mathbf{1} \to b \in F$. From the definition of a filter, it then follows that $(a \otimes \mathbf{1}) \to (a \otimes b) = a \to (a \otimes b) \in F$. Hence, by the property (ii) of a prefilter, $a \otimes b \in F$.

(b) By Lemma 2 (f) and Lemma 13 (a), $a \sim b \in F$ implies that $a \leftrightarrow b \in F$, which implies that $a \to b \in F$ and $b \to a \in F$. Hence, by item (a), the later implies that $a \overset{\circ}{\leftrightarrow} b \in F$, which implies (by applying Lemma 2 (f) again) that $a \sim b \in F$.

(c) The proof is direct from item (b) and the definition of a filter. □

Given a prefilter $F \subseteq E$, as usual, the following relation on E is an equivalence relation but not a congruence:

$$a \approx_F b \text{ iff } a \sim b \in F. \tag{24}$$

We shall denote by $[a]_F$ the equivalence class of $a \in E$ with respect to $\approx_F$, and by $E|F$ the quotient set associated with $\approx_F$.

Lemma 16
Let F *be a prefilter of a separated EQ-algebra* $\mathcal{E}$. *If* $a \approx_F b$ *and* $a' \approx_F b'$, *then* $a \to a' \in F$ *iff* $b \to b' \in F$.

PROOF: This follows directly from Lemma 14(c) and Lemma 13(b). □

Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$. By Lemma 14 (a), (b) and (c), $a \approx_F b$ and $a' \approx_F b'$imply $(a \wedge a') \approx_F (b \wedge b')$ and $(a \sim a') \approx_F (b \sim b')$.

Then one can define on $E|F$ the binary operations

$$[a]_F \wedge_F [b]_F = [a \wedge b]_F, \tag{25}$$
$$[a]_F \sim_F [b]_F = [a \sim b]_F, \tag{26}$$
$$[a]_F \rightarrow_F [b]_F = [a \rightarrow b]_F. \tag{27}$$

Moreover, by Lemma 14 (d), if $\mathcal{E}$ is an ℓEQ-algebra, then

$$[a]_F \vee_F [b]_F = [a \vee b]_F.$$

Also, one can define a binary relation $\leq_F$ on $E|F$ as follows:

$$[a]_F \leq_F [b]_F \text{ iff } [a]_F \wedge_F [b]_F = [a]_F. \tag{28}$$

Then, we have the following result.

Lemma 17
Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$.

(a) $[\mathbf{1}]_F = F$.

(b) $\mathcal{E}|F =< E|F, \wedge_F, \mathbf{1}_F >$ *is a meet-semilattice with the top element* $\mathbf{1}_F = F$.

(c) The binary relation $\leq_F$ given by (28) is a partial order on $E|F$ and satisfies

$$[a]_F \leq_F [b]_F \text{ iff } a \rightarrow b \in F \quad \textit{iff} \quad [a]_F \rightarrow_F [b]_F = [\mathbf{1}]_F. \tag{29}$$

PROOF: (a) From Lemma 13 (e), it then follows that $a \in F$ iff $a \in \{b \in E \mid b \sim \mathbf{1} \in F\} = [\mathbf{1}]_F$ and hence $F = [\mathbf{1}]_F$.

(b) Straightforward.

(c) From item (b), it follows that the binary relation $\leq_F$ given by (28) is a partial order. Also, we have the following chain of equivalences:

$[a]_F \leq_F [b]_F$ iff $[a \wedge b]_F = [a]_F$ iff $(a \wedge b) \sim a = a \rightarrow b \in F$ iff (from item (a)) $[\mathbf{1}]_F = [a \rightarrow b]_F = [a]_F \rightarrow_F [b]_F$ (from (27)). □

Theorem 6
Let F be a filter of a separated (ℓ)EQ-algebra $\mathcal{E}$. The factor algebra $\mathcal{E}|F = \langle E|F, \wedge_F, \otimes_F, \sim_F, F\rangle$ is a separated (ℓ)EQ-algebra (i.e., the relation $\approx_F$ given by (24) is a relative congruence of $\mathcal{E}$) and the mapping $f : a \longrightarrow [a]_F$ is a homomorphism of $\mathcal{E}$ onto $\mathcal{E}|F$.

PROOF: The fact that $\mathcal{E}|F$ is a EQ-algebra and f a homomorphism is an immediate consequence of Lemma 14, Lemma 15(c), Lemma 17(c), and Theorem 5. □

Theorem 6 can be also generalized to non-separated EQ-algebras. The definition of a filter in these algebras is the same as Definition 4 (see [39]). The problem, however, is that there are EQ-algebras with no non-trivial filter.

Theorem 7
Let $\mathcal{E}$ be a (ℓ)EQ-algebra $\mathcal{E}$. If there exists a filter F on $\mathcal{E}$, then the factor algebra $\mathcal{E}|F = \langle E|F, \wedge_F, \otimes_F, \sim_F, F\rangle$ is a separated (ℓ)EQ-algebra and the mapping $f : a \longrightarrow [a]_F$ is a homomorphism of $\mathcal{E}$ onto $\mathcal{E}|F$.

PROOF: We will only show that $\mathcal{E}|F$ is separated. Let $[a]_F \sim_F [b]_F = [\mathbf{1}]_F$. Then $[a \sim b]_F = [\mathbf{1}]_F$, i.e., $(a \sim b) \sim \mathbf{1} \in F$. Since also $\mathbf{1} \in F$, it follows from Lemma 13(b) that $a \sim b \in F$. This means that $a \approx_F b$, and hence, $[a]_F = [b]_F$. □

5. Good EQ-algebras

Proposition 5
The following are equivalent:

(a) An EQ-algebra $\mathcal{E}$ is good;

(b) $a \otimes (a \sim b) \leq b$ for all $a, b \in E$;

(c) $a \otimes (a \to b) \leq b$ for all $a, b \in E$;

(d) $(a \sim b) \otimes a \leq b$ for all $a, b \in E$;

(e) $(a \to b) \otimes a \leq b$ for all $a, b \in E$;

(f) $\mathbf{1} \to b = b$ for all $b \in E$.

PROOF: (a) iff (b) iff (c): The proofs are identical to those in [39].

(a) iff (d): By Lemma 6 (g), we get that (a) implies (d). Conversly, assume that (d) holds. Then $\tilde{b} = (\mathbf{1} \sim b) \otimes \mathbf{1} \leq b$. Lemma 2 (a) then implies that $\tilde{b} = b$, and hence, $\mathcal{E}$ is good.

(a) implies (e): It follows directly using Lemma 6 (h).

(e) implies (f): Assume that (e) holds. Then, by Lemma 2 (a), $b \leq 1 \to b = (\mathbf{1} \to b) \otimes \mathbf{1} \leq b$, i.e. $\mathbf{1} \to b = b$.

(f) implies (a): By (2), we have $b = \mathbf{1} \to b = (\mathbf{1} \wedge b) \sim \mathbf{1} = b \sim \mathbf{1}$. □

Lemma 18
The following properties hold in all good EQ-algebras for all $a, b, c \in E$:

(a) $a \leq (a \to b) \to b$,

(b) $(a \sim b) \otimes a \leq a \wedge b$ and $a \otimes (a \sim b) \leq a \wedge b$,

(c) $(a \to b) \otimes a \leq a \wedge b$ and $a \otimes (a \to b) \leq a \wedge b$,

(d) $a \leq b \to c$ implies $a \otimes b \leq c$ and $b \otimes a \leq c$.

PROOF: (a) By Lemma 5 (a), we have

$$a \le (a \sim (a \wedge b)) \sim (a \wedge b) \le (a \to b) \to (a \wedge b) \le (a \to b) \to b.$$

(b), (c) and (d) These properties follow immediately (by goodness) from Lemma 6 (g), (h) and (j), respectively. □

The following theorem shows that $\{\to, \mathbf{1}\}$-reducts of good EQ-algebras are *BCK-algebras* (for the definitions and basic properties of BCK-algebras, see [20, 28, 43, 45]). Thus, each good EQ-algebra can be viewed as a *BCK-meet-semilattice* with the extra operations $\otimes$ and $\sim$.

Theorem 8
The $\{\wedge, \to, \mathbf{1}\}$-reducts of good EQ-algebras are BCK-meet-semilattices*, where $\to$ is defined by (2).*

PROOF: By Lemma 5 (b), each good EQ-algebra $\mathcal{E}$ is separated. Hence, by Proposition 3 (b), $\mathcal{E}$ satisfies the quasi-identity: $a \to b = \mathbf{1}$ and $b \to a = \mathbf{1}$ implies $a = b$. Moreover, $a \to b = \mathbf{1}$ iff $a \wedge b = a$. Next, by Lemma 6 (f), we have $(a \to b) \to ((b \to c) \to (a \to c)) = \mathbf{1}$. Clearly, we have $a \to \mathbf{1} = \mathbf{1}$. Thus, from Lemma 18 (a) and Proposition 5 (vi) it follows that $\langle E, \wedge, \to, \mathbf{1}\rangle$ is a BCK-meet-semilattice. □

It is well known that BCK-algebras are exactly the $\{\to, \mathbf{1}\}$-subreducts of commutative integral residuated lattices (see [42, 15]), and residuated lattices are thus "hidden" inside. By Lemma 18 (d) and Theorem 8, it is easy to see that the multiplication $\otimes$ in a good EQ-algebra is less than or equal to the monoidal operation residing in the residuated lattice that corresponds to the BCK-algebra. As a consequence, the proof of the properties in the following lemma follows from the well-known results of the theory of BCK-algebras.

Lemma 19
The following properties hold in all good EQ-algebras:

(a) $a \le b \to c$ *iff* $b \le a \to c$;

(b) $a \to (b \to c) = b \to (a \to c)$ *(Exchange principle);*

(c) $a \to (b \to c) \le (a \otimes b) \to c$ *and* $a \to (b \to c) \le (b \otimes a) \to c$;

(d) For all indexed families $\{a_i\}_{i \in I}$ *in* E *we have*

$$\bigvee_{i \in I} a_i \to c = \bigvee_{i \in I} (a_i \to c),$$

provided that the supremum of $\{a_i\}_{i \in I}$ *exists in* E.

The following theorem demonstrates that adding the adjunction condition to EQ-algebra leads to a residuated EQ-algebra that is commutative. Moreover, it gives rise to a commutative residuated $\wedge$-semilattice (for the definition and basic properties of commutative residuated $\wedge$-semilattice, see [27]).

Theorem 9
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$ be a residuated EQ-algebra. Then its multiplication $\otimes$ is commutative. Moreover, $\mathcal{E}' = \langle E, \wedge, \otimes, \rightarrow, \mathbf{1} \rangle$ is a commutative residuated meet-semilattice, where $\rightarrow$ is defined by (2).

PROOF: Since each residuated EQ-algebra is good (by Lemma 5 (c)), it is a BCK-algebra (by Theorem 8). Moreover, it follows from the definition of a residuated EQ-algebra (see Definition 2 (11)) that the multiplication $\otimes$ coincides with the monoidal operation of the residuated lattice that corresponds to the BCK-algebra. The rest is obvious. $\square$

On can see that EQ-algebras (commutative or not) give rise to commutative residuated lattices. This is logically justified since the fuzzy equality $\sim$ remains symmetric, so there is only one implication to be derived from it — unlike the case of non-commutative residuated lattices, which have always two of them (and thus, a formal fuzzy logic based on such algebras must also have two implications). This suggests a possible further generalization of the concept of EQ-algebra by dropping the symmetry of the fuzzy equality $\sim$. As we shall show in a future article, if a non-symmetric fuzzy equality $\sim$ is allowed, then its implication $\rightarrow$ would have to be split into a right implication $\rightarrow_R$ and a left implication $\rightarrow_L$. Accordingly, in some of its occurrences, $\rightarrow$ would have to be replaced by $\rightarrow_R$, and in other ones, by $\rightarrow_L$. A similar splitting would take place in the fuzzy equality $\sim$, causing the whole algebra to be even more complicated. We prefer to postpone this added complexity, until an actual need for it materializes.

Lemma 20
Let $\mathcal{E}$ be a lattice-ordered and good EQ-algebra with the bottom element $\mathbf{0}$. The following holds for all $a, b, c \in E$:

(a) $a \sim b \leq (\neg a \wedge \neg c) \sim (\neg b \wedge \neg c) = \neg(a \vee c) \sim \neg(b \vee c)$;

(b) $((a \vee b) \sim c) \otimes (d \sim a) \leq \neg(d \vee b) \sim \neg c$;

(c) $(d \sim a) \otimes ((a \vee b) \sim c) \leq \neg(d \vee b) \sim \neg c$.

PROOF: (a) From Lemma 7 (b), Lemma 1 (c) and item (a), we get

$$a \sim b \leq \neg a \sim \neg b \leq (\neg a \wedge \neg c) \sim (\neg b \wedge \neg c) \leq \neg(a \vee c) \sim \neg(b \vee c).$$

(b) From Lemma 7 (b), item (a) and Axiom (A4), we obtain the following:

$$((a \vee b) \sim c) \otimes (d \sim a) \leq (\neg(a \vee b) \sim \neg c) \otimes (\neg d \sim \neg a) = \\ ((\neg a \wedge \neg b) \sim \neg c) \otimes (\neg d \sim \neg a) \leq (\neg d \wedge \neg b) \sim \neg c = \neg(d \vee b) \sim \neg c.$$

(c) This follows directly from item (b) and Theorem 3. $\square$

Proposition 6
The following properties are equivalent:

(a) an EQ-algebra $\mathcal{E}$ is residuated;

(b) the EQ-algebra $\mathcal{E}$ is good, and

$$(a \otimes b) \to c \leq a \to (b \to c)$$

holds for all $a, b, c \in E$;

(c) the EQ-algebra $\mathcal{E}$ is separated and

$$(a \otimes b) \to c = a \to (b \to c)$$

holds for all $a, b, c \in E$;

(d) the EQ-algebra $\mathcal{E}$ is good and

$$a \to b \leq (a \otimes c) \to (b \otimes c)$$

holds for all $a, b, c \in E$;

(e) the EQ-algebra $\mathcal{E}$ is good and

$$a \leq b \to (a \otimes b)$$

holds for all $a, b \in E$.

PROOF: (a) implies (b), (c), (d) and (e): By Theorem 9, Lemma 5 (c) and Lemma 19 (c).

(b) implies (c): This follows directly from Lemma 19(c) and the fact that every good EQ-algebra is separated.

(c) implies (a) and, therefore, it also implies (d) and (e).

(d) implies (e): Let us suppose that (d) holds. Then

$$a = \mathbf{1} \to a \leq (\mathbf{1} \otimes b) \to (a \otimes b) = b \to (a \otimes b).$$

(e) implies (a): Suppose (e) holds and assume that $a \otimes b \leq c$. Hence, from the assumption and the order properties of $\to$, we get $b \to c \geq b \to a \otimes b \geq a$. On the other hand, since $\mathcal{E}$ is good, $a \leq b \to c$ implies $a \otimes b \leq c$ (by Lemma 19 (c)). $\square$

Proposition 7
Let $\mathcal{E}$ be a good and complete EQ-algebra. Set

$$a \odot b = \bigwedge \{c \mid a \leq b \to c\}. \tag{30}$$

Then $\odot$ is a commutative, isotone w.r.t. $\leq$ in both arguments with a neutral element 1 and $\otimes \leq \odot$. Moreover, if $\to$ satisfies

$$a \to \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a \to b_j) \tag{31}$$

for all families $\{b_j\}_{j \in J}$ of E, then $\odot$ is associative, which means that $\mathcal{E}' = \langle E, \wedge, \vee, \odot, \to, \mathbf{1} \rangle$ is a complete residuated lattice and $\mathcal{E}'' = \langle E, \wedge, \vee, \odot, \leftrightarrow, \mathbf{1} \rangle$ is a complete residuated EQ-algebra.

PROOF: The monotonicity of $\odot$ follows immediately from the order properties of $\to$. Commutativity follows from Lemma 19(a). Also, $a \odot \mathbf{1} = \bigwedge\{c \mid a \leq c\} = a$.

Set $C = \{c \mid a \leq b \to c\}$. By Lemma 18(d), $a \otimes b \leq c$ holds for every $c \in C$, i.e, $a \otimes b \leq \bigwedge C = a \odot b$.

Now, if $\to$ satisfies (31), then, by Lemma 19 (d), $\to$ satisfies the identity

$$\bigvee_{i \in I} a_i \to \bigwedge_{j \in J} b_j = \bigwedge_{\substack{i \in I \\ j \in J}} (a \to b_j) \tag{32}$$

for all subfamilies $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ of E. The rest follows from known facts (see, e.g., [12, 35, 47])) using (32), the order properties of $\to$ (Lemma 2(g)), Proposition 3(b), Proposition 5 (f) and the Exchange principle (Lemma 19(b)). $\square$

Clearly, if $\mathcal{E}$ in Proposition 7 is residuated then $\otimes = \odot$.

6. $\mathbf{EQ_\Delta}$-algebras

The connective Δ is special and its variants appear in many non-classical logics (usually fulfilling some additional assumptions): it appears in the so-called symmetrical Heyting algebras (see [30]); it corresponds to the globalization in Intuitionistic logic (see [48]) and exponential in linear logic (see [46]), it has some properties of general modalities, and it is known as Baaz delta in fuzzy logics (see [5]).

In this Section, we will enrich good EQ-algebras with unary operation Δ fulfilling some additional assumptions as in the following definition:

Definition 5
An EQ_Δ-algebra is an algebra $\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \Delta, 0, 1\rangle$ which is a good EQ-algebra with bottom element 0 expanded by an unary operation $\Delta : E \longrightarrow E$ fulfilling the following axioms[3]:

(i) $\Delta 1 = 1$,

(ii) $\Delta a \leq a$,

(iii) $\Delta a \leq \Delta\Delta a$,

(iv) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$,

(v) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.

(vi) If $a \vee b$ and $\Delta a \vee \Delta b$ exist then $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,

[3]The Δ-axioms are from [36].

(vii) $\Delta a \vee \neg\Delta a = 1$. *(i.e., 1 is the unique upper bound in E of the set $\{\Delta a, \neg\Delta a\}$)*

Lemma 21
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. For all $a, b \in E$, it holds that

(a) *If $a \leq b$ then $\Delta a \leq \Delta b$.*

(b) $\Delta(a \to b) \leq \Delta a \to \Delta b$.

(c) *If $a \vee b$ and $\Delta a \vee \Delta b$ exist then $\Delta(a \vee b) = \Delta a \vee \Delta b$.*

(d) $\Delta a \to (\Delta a \to b) \leq \Delta a \to b$.

PROOF: (a) and (b) has been already proved in [36].

(c) Since $a, b \leq a \vee b$, then (by item (a)) $\Delta a \vee \Delta b \leq \Delta(a \vee b)$. Hence, by Definition 5 (vi), the result follows directly.

(d) By the order properties of $\to$ together with Lemma 18 (a) and Lemma 19 (a), it is easy to see that

$(\Delta a \to (\Delta a \to b)) \to (\Delta a \to b)$ is the upper bound in E of the set $\{\Delta a, \neg\Delta a\}$. Thus, by Definition 5 (vii), $(\Delta a \to (\Delta a \to b)) \to (\Delta a \to b) = 1$, that is $\Delta a \to (\Delta a \to b) \leq \Delta a \to b$. □

Our objective, in the rest of this paper, is devoted to characterize the representable class of EQ$_\Delta$-algebras. So a necessary modification for the definition of a (pre)filter on EQ$_\Delta$-algebra results by the following definition:

Definition 6
A prefilter *F on an EQ$_\Delta$-algebra $\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \Delta, 0, 1\rangle$ is a prefilter on its good EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ that satisfies, for all $a \in E$, $\Delta a \in F$ whenever $a \in F$. Filters and prime (pre)filters are understood similarly.*

Trivially, the singleton $\{1\}$ is a filter in any EQ$_\Delta$-algebra, and is contained in any other filter. Note that if F is prime and Q is a prefilter such that $F \subseteq Q$, then Q is a prime prefilter.

We extend to EQ$_\Delta$-algebras the following two results, proved by El-Zekey in the setting of good EQ-algebras (see [13, Lemma 13 and Proposition 8], respectively, for the analogous results). The proofs are completely the same as El-Zekey's proofs.

Proposition 8
For a prefilter F of an EQ$_\Delta$-algebra $\mathcal{E}_\Delta$, the following properties are equivalent:

(i) *F is a filter.*

(ii) *For all $b, c \in E$, $b \in F$ implies $c \to (b \otimes c) \in F$ and $c \to (c \otimes b) \in F$.*

(iii) *For all $b, c, d \in E$, $b \in F$ implies $d \to (d \otimes (c \to (b \otimes c))) \in F$.*

Proposition 9
Let F be a prefilter of a prelinear EQ$_\Delta$-algebra $\mathcal{E}_\Delta$. Then the following properties are equivalent

(i) F *is prime,*

(ii) *for each* $a, b \in E$ *such that* $a \vee b \in F$, $a \in F$ *or* $b \in F$,

(iii) *for each* $a, b \in E$ *such that* $a \vee b = 1$, $a \in F$ *or* $b \in F$,

(iv) E/F *is a chain, i.e. is linearly (totally) ordered by* $\leq_F$.

It has been shown that (see Section 4) if F is a prefilter of a good EQ-algebra $\mathcal{E}$, then all the operations of $\mathcal{E}$ except for the multiplication are compatible with the equivalence relation $\approx_F$ given by (24). If F is a filter, then $\approx_F$ is a congruence and the quotient algebra $\mathcal{E}/F$ is a good EQ-algebra. So, for a filter F on EQ$_\Delta$-algebra $\mathcal{E}_\Delta$, it is easy to see that Δ is compatible with $\approx_F$ (since $a \sim b \in F$ implies $\Delta(a \sim b) \in F$ and hence $\Delta a \sim \Delta b \in F$). Thus $\mathcal{E}_\Delta/F$ is an EQ$_\Delta$-algebra.

Lemma 22
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. For any congruences θ and ϕ of $\mathcal{E}_\Delta$, we have

(i) $F = [1]_\theta = \{a \in E : a\theta 1\}$ *is a filter of* $\mathcal{E}_\Delta$.

(ii) $a\theta b$ *iff* $(a \sim b)\theta 1$ *iff* $(a \to b)\theta 1$ *and* $(b \to a)\theta 1$ *iff* $(a \leftrightarrow b)\theta 1$ *iff* $(a \overset{\circ}{\leftrightarrow} b)\theta 1$.

(iii) $[1]_\theta = [1]_\phi$ *implies* $\theta = \phi$.

(iv) $[1]_\theta = \{1\}$ *iff* θ *is the trivial congruence.*

PROOF: (i) It is already proved in [13, Lemma 12 (i)] that $F = [1]_\theta$ is a filter of $\mathcal{E}$. So, it remains only to show that $\Delta a \in [1]_\theta$ whenever $a \in [1]_\theta$. Suppose that $a \in [1]_\theta$. Then $a\theta 1$ and hence $\Delta a \theta \Delta 1$ (since θ is a congruence). Since $\Delta 1 = 1$, $\Delta a \theta 1$ and hence $\Delta a \in [1]_\theta$.

(ii), (iii) and (iv) The analogous results in good EQ-algebra has been established by El-Zekey [13]. The proof is valid in the present setting and applies verbatim here for the EQ$_\Delta$-algebra $\mathcal{E}_\Delta$. $\square$

A collection of all filters of an EQ$_\Delta$-algebra $\mathcal{E}_\Delta$ will be denoted by $\mathcal{F}(\mathcal{E}_\Delta)$. Note that the first author (see [13]) has shown that, for any separated EQ-algebra $\mathcal{E}$, the lattice $\mathcal{F}(\mathcal{E})$ of filters of $\mathcal{E}$ is isomorphic to the lattice of relative congruences of $\mathcal{E}$, via the mutually inverse maps $F \longmapsto \approx_F$ and $\theta \longmapsto [\mathbf{1}]_\theta$. Consequently, in the case of a good EQ-algebra $\mathcal{E}$, the lattice of filters $\mathcal{F}(\mathcal{E})$ is in bijective correspondence with the lattice of congruences $\mathbf{Con}(\mathcal{E})$. This is easily extended to any EQ$_\Delta$-algebra $\mathcal{E}_\Delta$ in the following theorem.

Theorem 10
For any EQ$_\Delta$-algebra $\mathcal{E}_\Delta$, the lattice $\mathcal{F}(\mathcal{E}_\Delta)$ of filters of $\mathcal{E}$ is isomorphic to the lattice $\mathbf{Con}(\mathcal{E}_\Delta)$ of congruences of $\mathcal{E}_\Delta$, via the mutually inverse maps $F \longmapsto \approx_F$ and $\theta \longmapsto [1]_\theta$.

PROOF: Just note that, by Lemma 22 (i), $[1]_\theta$ is a filter of $\mathcal{E}_\Delta$. Moreover, the relation $\approx_F$ given by (24) is a congruence relation on $\mathcal{E}_\Delta$. Hence, using Lemma 22 (iii), the proof proceeds in a standard way (cf. [13], Theorem 9). $\square$

7. Representable $\mathbf{EQ_\Delta}$-algebras

Recall that an algebra which is a subdirect product of those with underlying linear order is said to be *representable*. We devote this section to characterization of the representable class of EQ_Δ-algebras, along the lines parallel to El-Zekey's characterization of representable good EQ-algebras [13].

The set-subtraction of set Y from set X will be denoted $X - Y$. Recall that an element b of a lattice is *meet-irreducible* if, for any finite set X, $\bigwedge X = b$ implies $b \in X$. If this property holds for all sets X, we call b *completely meet-irreducible*.

The set of all prefilters of an EQ_Δ-algebra $\mathcal{E}_\Delta$ will be denoted by $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$. For every $X \subseteq E$, the smallest prefilter of $\mathcal{E}_\Delta$ containing X, i.e. the intersection of all prefilters $F \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$ such that $X \subseteq F$, is said to be the *prefilter generated by* X and will be denoted by $\langle X \rangle$. For each $a \in E$, we abbreviate $\langle \{a\} \rangle$ by $\langle a \rangle$. Obviously, if X is a filter then $\langle X \rangle = X$. It is clear that if $X_1 \subseteq X_2$ then $\langle X_1 \rangle \subseteq \langle X_2 \rangle$. If $X = Y \cup \{a\}$, we will write $\langle Y, a \rangle$ for $\langle X \rangle$. The set of nonnegative integers will be denoted by ω. For $a, b \in E$ and $n \in \omega$, $a^n \to b$ is defined as follows: $a^0 \to b = b$ and $a^{n+1} \to b = a \to (a^n \to b)$.

It is well known that the *Delta Deduction Theorem (DT_Δ)* holds for fuzzy logics expanding BCK logic (cf. [10]). Recall that (see Theorem 8), when considering implication only, the corresponding reduct of EQ-algebras are BCK-algebras. Hence, the following two results can be considered as the algebraic counterpart of DT_Δ. However, we believe their algebraic proofs should be presented to the interested reader, as we here do.

Proposition 10
Let $\mathcal{E}_\Delta$ be an EQ_Δ-algebra. Then $\langle \emptyset \rangle = \{1\}$, and for every $\emptyset \neq X \subseteq E$, we have

$$\langle X \rangle = \{a \in E : \Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_n \to a)...)) = 1 \quad \text{for some distinct elements } b_1, ..., b_n \in X,\ n \in \omega\}. \tag{33}$$

In particular, for every $b \in E$

$$\langle b \rangle = \{a \in E : \Delta b \to a = 1\}. \tag{34}$$

PROOF: Trivially, $\langle \emptyset \rangle = \{1\}$. Let X be a non-empty subset and let M denote the right-hand side above. It is clear that $1 \in M$. To see that M is a prefilter assume $a \in M$ and $a \to b \in M$, i.e., $\Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to a)...)) = 1$ for some distinct elements $b_1, ..., b_m \in E$, $m \in \omega$ and $\Delta c_1 \to (\Delta c_2 \to (... \to (\Delta c_n \to (a \to b))...)) = 1$ for some distinct elements $c_1, ..., c_n \in E$, $n \in \omega$.

Then, by EP of $\to$, $a \to (\Delta c_1 \to (\Delta c_2 \to (... \to (\Delta c_n \to b))...)) = 1$, thus $a \leq \Delta c_1 \to (\Delta c_2 \to (... \to (\Delta c_n \to b))...)$, which yields, by the order properties of $\to$, $1 = \Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to a)...)) \leq \Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to (\Delta c_1 \to (\Delta c_2 \to (... \to (\Delta c_n \to b))...)))...))$. If not all b_i's and c_j's are distinct (i.e., $b_i = c_j$ for some i, j) then use repeated applications of Lemma 21 (d) together with EP of $\to$ to eliminate the repetation. Since all b_i's and c_j's belong to X, it follows that $b \in M$. It remains to prove that $\Delta a \in M$ whenever $a \in M$. Assume $a \in M$, i.e., $\Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to a)...)) = 1$ for some $b_1, ..., b_m \in E$ and $m \in \omega$. Then, by Definition 5 (i), Lemma 21 (b) and the order properties of $\to$, $1 = \Delta 1 = \Delta(\Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to a)...))) \leq \Delta\Delta b_1 \to (\Delta\Delta b_2 \to (... \to (\Delta\Delta b_m \to \Delta a)...))$. Then use repeated applications of Definition 5 (iii) and EP of $\to$, we obtain $\Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_m \to \Delta a)...)) = 1$, it follows that $\Delta a \in M$. We have proved $M \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$. Moreover, one readily sees that (i) $X \subseteq M$, and (ii) $M \subseteq F$ whenever $X \subseteq F$ for $F \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$, so that $M = \langle X \rangle$ as desired. □

Corollary 1
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra, $F \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$, and $a \in E$. Then

$$\langle F, a \rangle = \{b \in E : \Delta a \to b \in F\}\,.$$

PROOF: Obviously, $\Delta a \in \langle F, a \rangle$ (since $\langle F, a \rangle$ is a prefilter of $\mathcal{E}_\Delta$ and $a \in \langle F, a \rangle$). If $\Delta a \to b \in F$ then $\Delta a \to b \in \langle F, a \rangle$ and hence $b \in \langle F, a \rangle$. Conversely, suppose that $b \in \langle F, a \rangle$. In virtue of Proposition 10, there exist $k \in \omega$ and distinct elements $z_1, ..., z_k \in F \cup \{a\}$ such that $\Delta z_1 \to (.... \to (\Delta z_k \to b)...) = 1$. Note that among the z_i's, there are at most one a and for all $z_j \in F$, it follows that $\Delta z_j \in F$ (since F is aprefilter). Since $\Delta z_1 \to (.... \to (\Delta z_k \to b)...) = 1 \in F$, it follows that either $b \in F$ and hence $\Delta a \to b \in F$ (by the fact that $b \leq \Delta a \to b$) or $\Delta a \to b \in F$. This shows that $\Delta a \to b \in F$ iff $b \in \langle F, a \rangle$ and completes the proof. □

Proposition 11
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra, $F \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$, and $a, b \in E$. If $a \vee b$ and $\Delta a \vee \Delta b$ exist then $\langle F, a \vee b \rangle = \langle F, a \rangle \cap \langle F, b \rangle$.

PROOF: It is straightforward to show that if $x \leq y$ then $\langle F, y \rangle \subseteq \langle F, x \rangle$. Therefore $\langle F, a \vee b \rangle \subseteq \langle F, a \rangle \cap \langle F, b \rangle$. Conversely, if $x \in \langle F, a \rangle \cap \langle F, b \rangle$ then $\Delta a \to x \in F$ and $\Delta b \to x \in F$. Hence, by Lemma 19 (d), $(\Delta a \vee \Delta b) \to x = (\Delta a \to x) \wedge (\Delta b \to x) \in F$. Lemma 21 (c) leads us to state that $\Delta(a \vee b) \to x \in F$, and so $x \in \langle F, a \vee b \rangle$. □

The set of all prefilters of an EQ$_\Delta$-algebra $\mathcal{E}_\Delta$ is closed under intersection and so forms a complete lattice under inclusion. For every family $\{F_i\}_{i \in I}$ of prefilters of $\mathcal{E}$, we have that $\bigwedge_{i \in I} F_i = \bigcap\limits_{i \in I} F_i$ and $\bigvee_{i \in I} F_i = \left\langle \bigcup\limits_{i \in I} F_i \right\rangle$.

The proof of the following proposition is somewhat similar to Pałasinski's proof in [44] of the distributivity of filters in BCK-algebras.

Proposition 12
Let $\mathcal{E}_\Delta$ be an EQ_Δ-algebra. Then $(\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta), \subseteq)$ is a complete distributive lattice. More precisely, for any set $\{Q_j | j \in J\}$ of prefilters and any prefilter F of $\mathcal{E}_\Delta$,

$$F \cap \bigvee_j Q_j = \bigvee_j \{F \cap Q_j | j \in J\}.$$

PROOF: We need only show that $F \cap \bigvee_j Q_j \subseteq \bigvee_j (F \cap Q_j)$, so let $a \in F \cap \bigvee_j Q_j$. Since $\bigvee_j Q_j = \langle \cup \{Q_j | j \in J\}\rangle$, there exist $b_1, ..., b_n \in \cup\{Q_j | j \in J\}$ such that $\Delta b_1 \to (\Delta b_2 \to (... \to (\Delta b_n \to a)...)) = 1$. For each $k \in \{1, ..., n\}$, $b_k \in Q_{\pi(k)}$ for some $\pi(k) \in J$.

Set $c_1 = \Delta b_2 \to (... \to (\Delta b_n \to a)...)$. Then $\Delta b_1 \leq c_1$ (by separation). Since $b_1 \in Q_{\pi(1)}$ and $Q_{\pi(1)}$ is prefilter then $\Delta b_1 \in Q_{\pi(1)}$ and hence $c_1 \in Q_{\pi(1)}$. For each $k \in \{1, ..., n-1\}$, we recursively define

$$c_{k+1} = \Delta b_{k+2} \to (... \to (\Delta b_n \to (c_k \to ...(c_1 \to a)...))).$$

Expanding c_k in the above expression and then applying Lemma 19 (b) gives

$$\begin{aligned} c_{k+1} = [\Delta b_{k+1} &\to (\Delta b_{k+2} \to ... \to (\Delta b_n \to (c_{k-1} \to ...(c_1 \to a)...)] \\ &\to [\Delta b_{k+2} \to ... \to (\Delta b_n \to (c_{k-1} \to ...(c_1 \to a)...]. \end{aligned}$$

Now, by Lemma 18 (a), we see that $\Delta b_{k+1} \leq c_{k+1}$. Thus, $c_k \in Q_{\pi(k)}$ for each $k \in \{1, ..., n\}$. Clearly, by Lemma 2 (d), $c_k \geq a \in F$ so $c_k \in F \cap Q_{\pi(k)}$. By definition, $c_n = c_{n-1} \to (... \to (c_1 \to a)...)$, hence (by Lemma 21 (ii), the order properties of $\to$ and Definition 5 (ii)) $\Delta c_n = \Delta[c_{n-1} \to (... \to (c_1 \to a)...)] \leq \Delta c_{n-1} \to (... \to (\Delta c_1 \to \Delta a)...) \leq \Delta c_{n-1} \to (... \to (\Delta c_1 \to a)...)$. Hence, $\Delta c_n \to (\Delta c_{n-1} \to (... \to (\Delta c_1 \to a)...)) = 1$, so $a \in \langle \cup\{F \cap Q_k | k \in \{1, ..., n\}\}\rangle \subseteq \bigvee_j \{F \cap Q_j | j \in J\}$. □

Lemma 23
Let F be a prefilter of a prelinear EQ_Δ-algebra $\mathcal{E}_\Delta$. Then the following properties are equivalent:

(i) F is prime.

(ii) $\{Q \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta) : F \subseteq Q\}$ is linearly ordered under inclusion.

(iii) F is meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$.

PROOF: (i) $\Longrightarrow$ (ii): Suppose that F is a prime prefilter of $\mathcal{E}_\Delta$. Then F is a prime prefilter of $\mathcal{E}$ and hence, see [13, Lemma 12 (i)], $\{Q \in \mathbf{P}\mathcal{F}(\mathcal{E}) : F \subseteq Q\}$ is linearly ordered under inclusion. Since $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta) \subseteq \mathbf{P}\mathcal{F}(\mathcal{E})$, the result follows immediately.

(ii) $\Longrightarrow$ (iii): Let $Q, R \in \mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$. If $Q \cap R = F$, then $F \subseteq Q, R$. So, by item (ii) and without loss of generality, $Q \subseteq R$, hence $Q = F$.

(iii) $\Longrightarrow$ (i): Suppose $a \vee b \in F$. Then, $F = \langle F, a \vee b\rangle = \langle F, a\rangle \cap \langle F, b\rangle$ (by Proposition 11). By meet-irreducibility, $F = \langle F, a\rangle$ or $F = \langle F, b\rangle$, so $a \in F$ or $b \in F$. □

Lemma 24
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. Then

(i) Every meet-irreducible prefilter of $\mathcal{E}_\Delta$ contains a minimal meet-irreducible prefilter.

(ii) $\bigcap\{F : F$ is a minimal meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)\} = \{1\}$.

PROOF: (i) Since $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$ is a complete distributive lattice (see Proposition 12), (i) follows from the result by van Alten (see [3, Lemma 2.7]) about complete distributive lattices.

(ii) It is already proved that (see [13, Lemma 13]), for each separated EQ-algebra $\mathcal{E}$ (and hence for each good), $\bigcap\{F : F$ is meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E})\} = \{1\}$. Hence, (ii) now follows from item (i) with the observation that $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta) \subseteq \mathbf{P}\mathcal{F}(\mathcal{E})$ and noting that $\{1\}$ is a prefilter of $\mathcal{E}_\Delta$ and is contained in any other prefilter. □

A nonempty downward closed subset $I \subseteq E$ is called an *ideal* of a lattice-ordered EQ-algebra $\mathcal{E}$ if it is closed under finite joins. For each $a \in E$, set $F_a = \{b \in E : a \vee b = 1\}$. We shall use $[X)$ and $(X]$ to denote the *upward* and *downward closures*, respectively, of a subset X of a partially ordered set.

Lemma 25
Let $\mathcal{E}_\Delta$ be a lattice-ordered EQ$_\Delta$-algebra. Then, for each $a \in \mathcal{E}_\Delta$, F_a is a prefilter of $\mathcal{E}_\Delta$. Moreover, if I is an ideal of $\mathcal{E}_\Delta$, then $I' = \bigcup\{F_a : a \in I\}$ is a prefilter of $\mathcal{E}_\Delta$.

PROOF: It is already proved that (see [13, Lemma 15]), for each separated lattice-ordered EQ-algebra $\mathcal{E}$ (and hence for each good lattice-ordered EQ-algebra), I' is a prefilter of $\mathcal{E}$. So it remains only to show that if $b \in I'$, then $\Delta b \in I'$. Therefore, suppose that $b \in I'$. Then for some $a \in I$, we have $a \vee b = 1$. Hence, by Lemma 21 (c) and Definition 5 (i), $\Delta a \vee \Delta b = \Delta(a \vee b) = \Delta 1 = 1$. Hence, by Definition 5 (ii), $1 = \Delta a \vee \Delta b \leq a \vee \Delta b$, that is $\Delta b \in I'$. The first part of the lemma now follows from the observation that $(a]$ is an ideal and $F_a = (a]'$. □

Lemma 26
Let F be a prefilter of a prelinear EQ$_\Delta$-algebra $\mathcal{E}_\Delta$. Then F is a minimal prime prefilter of $\mathcal{E}_\Delta$ iff $F = \bigcup\{F_a : a \in E - F\}$.

PROOF: The analogous result in biresiduated lattices has been established by C. J. van Alten [3, Lemma 3.4]. The machinery employed in his proof consists of Lemma 25 and Proposition 9. So, his proof is valid in the present setting and applies verbatim here for prelinear EQ$_\Delta$-algebra $\mathcal{E}_\Delta$. □

Accordingly, using Lemma 25 and Lemma 26 together with Proposition 1 and condition (iii) of Lemma 23, we easily extend to EQ$_\Delta$-algebras the following result, proved by El-Zekey [13, Lemma 17] in the setting of good EQ-algebras. The proof is completely the same as El-Zekey's proof.

Lemma 27
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. If $\mathcal{E}_\Delta$ satisfies (19), or equivalently (20), then for each $a \in E$, F_a is a filter of $\mathcal{E}_\Delta$ and, if I is an ideal of $\mathcal{E}_\Delta$, then $\bigcup\{F_a : a \in I\}$ is a filter of $\mathcal{E}_\Delta$. Thus, every minimal prime prefilter of $\mathcal{E}_\Delta$ is a filter.

We have settled all the auxiliary results, so we can prove that the characterization theorem obtained till now for representable good EQ-algebras (see [13]) hold also for EQ$_\Delta$-algebras. The proof is completely the same as El-Zekey's proof. We shall supply the proof because of the importance of the statement and to make the paper self-contained:

Theorem 11
Let $\mathcal{E}_\Delta$ be an EQ$_\Delta$-algebra. The following statements are equivalent:

(i) $\mathcal{E}_\Delta$ is subdirectly embeddable into a product of linearly ordered good EQ$_\Delta$-algebras; i.e., $\mathcal{E}_\Delta$ is representable.

(ii) $\mathcal{E}_\Delta$ satisfies (19), or equivalently (20).

(iii) $\mathcal{E}_\Delta$ is prelinear and every minimal prime prefilter of $\mathcal{E}_\Delta$ is a filter of $\mathcal{E}_\Delta$.

PROOF: (i) $\Longrightarrow$ (ii) It is obvious that if $\mathcal{E}_\Delta$ is representable then it satisfies the identity (19), or equivalently (20) (since in linearly ordered EQ-algebra one has either $x \to y = 1$ or $y \to x = 1$ for all x, y).

(ii) $\Longrightarrow$ (iii): By Proposition 1, $\mathcal{E}_\Delta$ is prelinear and hence it follows from Lemma 27 that every minimal prime prefilter of $\mathcal{E}_\Delta$ is a filter of $\mathcal{E}_\Delta$.

(iii) $\Longrightarrow$ (i) Since $\mathcal{E}$ is prelinear, Lemma 23 holds for $\mathcal{E}_\Delta$, hence the prime prefilters of $\mathcal{E}_\Delta$ are precisely the meet-irreducible elements of $\mathbf{P}\mathcal{F}(\mathcal{E}_\Delta)$. Let X be the set of all minimal prime prefilters of $\mathcal{E}_\Delta$. By Lemma 24 (ii), $\bigcap X = \{1\}$, hence, by Theorem 10, $\bigcap\{\approx_F : F \in X\}$ is the trivial congruence. Thus, by standard techniques of universal algebra (Cf. [8]), the natural homomorphism $h : \mathcal{E}_\Delta \longrightarrow \prod\limits_{F \in X} (\mathcal{E}_\Delta / \approx_F)$ defined by $h(a) = \langle [a]_F \rangle_{F \in X}$ is a subdirect embedding of $\mathcal{E}$ into a direct product of $\{\mathcal{E}_\Delta / \approx_F : F \in X\}$. Using Proposition 9, $\mathcal{E}_\Delta / \approx_F$ is linearly ordered EQ$_\Delta$-algebra for each $F \in X$, which completes the proof. $\Box$

By Proposition 2, we get the following corollary as a simple and an alternative characterization of the one exist in Theorem 11 for the class of representable commutative EQ$_\Delta$-algebra:

Corollary 2
Let $\mathcal{E}_\Delta$ be a commutative EQ$_\Delta$-algebra. The following statements are equivalent:

(i) $\mathcal{E}_\Delta$ is representable.

(ii) $\mathcal{E}_\Delta$ satisfies (23).

(iii) $\mathcal{E}_\Delta$ is prelinear and every minimal prime prefilter of $\mathcal{E}_\Delta$ is a filter of $\mathcal{E}_\Delta$.

8. Conclusion

In this paper, we have continued the study of EQ-algebras and their special cases, initiated in [38, 39] and [13]. We showed that the commutativity axiom of the multiplication originally assumed in [39, Definition 1] is superfluously restrictive, i.e., a weaker requirement put on non-commutative multiplication is sufficient to guarantee all the expected general properties of fuzzy equalities and EQ-algebras. This opens an exciting possibility to develop a fuzzy logic with a non-commutative conjunction but a single implication only (see [40]). We have also proved several important properties of EQ-algebras and their special cases. A great importance has been given to the study of good EQ-algebras. Our studies show that the "goodness" property has important consequences, e.g., $\{\rightarrow, 1\}$-reducts of good EQ-algebras are BCK-algebras. This fact played an important role in the characterization of the representable class of good EQ-algebras (see [13]). We enriched good EQ-algebras with an unary operation Δ (the so-called Baaz delta). We showed that the characterization theorem obtained till now for representable good EQ-algebras (see [13]) holds also for the enriched algebras.

The following questions can be raised: what is the role of non-separated EQ-algebras and whether we need them. The original idea when introducing EQ-algebras was to extract the fundamental properties of fuzzy equality with respect to the basic structure — the $\wedge$-semilattice, as mentioned in the introduction. The consequence is that two different elements can still be "fuzzy equal" in the degree $\mathbf{1}$. The development of EQ-logic in [40, 36] (see also [13]) has revealed, however, that the "goodness" property (and thus also separateness) is necessary for a reasonably behaving logic. Therefore, the pure (non-separated) EQ-algebras are quite specific algebraic structures that might give us answer about the really necessary properties of (fuzzy) equality; for example, why separateness is necessary. It seems to us too early to answer the questions posed above and we leave them open in this paper.

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